Economics 3010
Topics in Macroeconomics 3
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Lecture 8: Introduction to asset pricing

1 Facts

According to Aiyagari (1993), the average annual real rate of return on 3-month U.S. Treasury bills in the post-war period has been about 1 percent. On stocks, this rate of return has been about 6 percent (4 percent in the last 200 years). Thus the equity premium was about 5 percentage points. According to Jagannathan et al. (2000), the equity premium was about 7 percentage points 1926-70, but only 0.7 of a percentage point 1970-99.

This set of lecture notes will develop the consumption-based capital asset pricing model (C-CAPM) which is based on the concept of competitive equilibrium. A subsequent set of lecture notes will investigate if it can account for the above facts. It can’t—unless we focus on the last 30 years only. But we will discuss how the model might be extended to account for these facts.
2 Prequel to C-CAPM: no-arbitrage pricing

2.1 Motivational story

(This example is modified from Björk (1998).)

Consider the UK based company F&H Ltd which today \((t = 0)\) has signed a contract with an American counterpart, ACME Inc. The contract stipulates that ACME will deliver 150 laptop computers to F&H exactly six months from now \((t = 1)\). Furthermore it is stipulated that F&H will pay 1000 US dollars per computer to ACME at the time of delivery \((t = 1)\). For the sake of argument will assume that the present spot currency exchange rate between sterling and US dollars is $1.5 per £.

One of the problems with this contract from the point of view of F&H is that it involves a considerable currency risk. Since F&H don’t know what the exchange rate will be six months from now, this means that they don’t know how many £ they will have to pay at \(t = 1\). Thus F&H face the problem of how to hedge itself against this currency risk, and we now list a number of natural strategies.

1. Buy $150000 today for £100000. While this certainly eliminates the currency risk, it also ties up a substantial amount of money for a considerable period of time. Even more seriously, F&H might not have £100000 now and is not able to borrow it either.

2. Another arrangement that doesn’t involve any outlays at all today is that F&H go to the forward market and signs a forward contract for $150000 with delivery six months from now. Such a contract may be negotiated with a commercial bank and it stipulates

   - The bank will, at \(t = 1\), deliver $150000 to F&H.
   - F&H will, at \(t = 1\), pay for this delivery at the rate of £\(K\) per $. The
exchange rate $K$, which is called the forward rate at $t = 0$ for delivery at $t = 1$, is determined at $t = 0$. By definition, the cost of entering into the contract is zero, and the forward rate $K$ is determined by supply and demand in the forward market. Notice, however, that even if the price of entering the forward contract at $t = 0$ is zero, the contract may very well fetch a non-zero price during the interval $[0, 1]$, say in three months’ time.

Let us now assume that the forward rate today for delivery in six months equals £0.70 per $. If F&H enter the forward contract this means that there are no outlays today, and in six months it will get $150000 at £105000. Since the forward rate is determined today, F&H have again eliminated the currency risk.

However, a forward contract also has some drawbacks, which are related to the fact that it is a mutually binding contract. To see this let’s consider two scenarios.

- Suppose the spot exchange rate at $t = 1$ turns out to be £0.80 per $. Then F&H can pat themselves on the back for having saved £15000.
- Suppose on the other hand that the spot exchange rate at $t = 1$ turns out to be 0.60. Then F&H are forced to buy dollars at the rate 0.70 despite the fact that the going market rate is 0.60, representing a missed opportunity to save money.

3. What F&H would like is of course a contract which hedges them against a high spot price of dollars at $t = 1$ while still allowing it to take advantage of a low spot rate at $t = 1$. Such contracts do in fact exist, and they are called options.

**Definition** A (European) call option on the amount of $x$ dollars with strike price $K$ pounds per dollars and exercise date $T$ is a contract with the following properties.

- The holder of the contract has (at $t = T$ only) the right to buy $x$ dollars at the price $K$ pounds per dollar.
• The holder of the option has no obligation to buy the dollars.

What is such an option worth, setting \( x = 1 \) and \( K = 0.70 \)? Well, suppose there is a market for loans/deposits where £1 today (\( t = 0 \)) pays £1.02 at \( t = 1 \). Suppose also that there are precisely two possibilities, each equally likely. Either the spot dollar rate at \( t = 1 \) is 0.60 or it’s 0.80.

It’s tempting at this point (isn’t it?) to suggest that the “correct” price of the option is its discounted expected payoff. Its payoff if the spot rate is 0.60 is zero; its payoff when the spot rate is 0.80 is £0.10. Multiplying by the probability of this happening, we get 0.05. Discounting back to \( t = 0 \), we get an option price of about £0.049.

Another tempting idea is to say that there is no such thing as the “correct” price of the option, it’s determined by supply and demand.

Actually, both these apparently plausible answers are wrong, in the sense that they would create an arbitrage opportunity. By using a no-arbitrage argument, we can find the really correct price. By following these lecture notes, you will, at the end, be able to go back and compute the correct price of that option.

How far can we get in determining the “correct” price of assets by just assuming that there are no arbitrage opportunities? Notice that this is a weaker assumption than (competitive) equilibrium. Surprisingly far, as it turns out. In particular, we can work out how to price derivatives—claims that are defined in terms of other securities, e.g. options.
2.2 A two-period model

Let \( t = 0, 1 \) (today and tomorrow). There are \( N \) securities. The price of security \( n \) at time \( t \) is denoted by \( S_t^n \). We write

\[
S_t = \begin{bmatrix}
S_1^1 \\
S_2^2 \\
\vdots \\
S_N^N
\end{bmatrix}.
\]

\( S_0 \) is deterministic, but \( S_1 \) is stochastic. We write

\[
S_1(\omega) = \begin{bmatrix}
S_1^1(\omega) \\
S_2^2(\omega) \\
\vdots \\
S_N^N(\omega)
\end{bmatrix}
\]

where \( \omega \in \Omega = \{\omega_1, \omega_2, \ldots, \omega_M\} \).

Now define the matrix \( D \) via

\[
D_{N \times M} = \begin{bmatrix}
S_1^1(\omega_1) & S_1^1(\omega_2) & \ldots & S_1^1(\omega_M) \\
S_2^2(\omega_1) & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & \ddots \\
S_N^N(\omega_1) & \ldots & \ldots & S_N^N(\omega_M)
\end{bmatrix} = \begin{bmatrix}
d_1^1 & d_1^2 & \ldots & d_1^M \\
d_2^1 & d_2^2 & \ldots & d_2^M \\
\vdots & \vdots & \ddots & \vdots \\
d_M^1 & d_M^2 & \ldots & d_M^M
\end{bmatrix}.
\]

**Definition.** The set of vectors of real numbers \( x = (x_1, x_2, \ldots, x_n) \) is denoted by \( \mathbb{R}^n \).

When we want to say that a vector \( x \) is a member of \( \mathbb{R}^n \) we write \( x \in \mathbb{R}^n \). What this means is simply that \( x \) is a vector consisting of \( n \) real numbers.

**Definition.** The set of vectors of positive real numbers \( x = (x_1, x_2, \ldots, x_n) \) is denoted by \( \mathbb{R}^n_+ \). A real number \( \alpha \) is said to be positive if \( \alpha \geq 0 \).

**Definition.** A portfolio is a vector \( h \in \mathbb{R}^N \). Interpretation: \( h_n \) is the number of securities of type \( n \) purchased at \( t = 0 \).
Remark: Fractional holdings as well as short positions \( h_n < 0 \) are allowed.

The *value* of a portfolio \( h \) at time \( t \) is given by

\[
V_t(h) = \sum_{n=1}^{N} h_n S_t^n = h \cdot S_t
\]

where it should be apparent that \( \cdot \) represents the (Euclidean) inner product (scalar product).

**Definition.** A vector \( h \in \mathbb{R}^N \) is called an *arbitrage portfolio* if

\[
V_0(h) < 0
\]

and

\[
V_1(h) \geq 0
\]

for all \( \omega \in \Omega \).

Remark. We can weaken the condition “for all \( \omega \in \Omega \)” to “with probability one” if we want — but in this context that wouldn’t add much.

**Theorem.** Let securities prices \( S \) be as above. Then there exists no arbitrage portfolio *iff* there exists a \( z \in \mathbb{R}_+^M \) such that

\[
S_0 = Dz.
\]

Remark. This means that today’s (period 0’s) price vector has to lie in the convex cone spanned by tomorrow’s (period 1’s) possible prices vectors. (A *convex cone* is a subset \( C \) of a vector space \( X \) such that for any \( x, y \in C \) and \( \alpha \geq 0 \) we have \( \alpha x \in C \) and \((x + y) \in C \).)

**Proof.** Absence of arbitrage opportunities means that the following system of inequalities has no solution for \( h \).

\[
\begin{cases}
    h \cdot S_0 < 0 \\
    h \cdot d^j \geq 0 \text{ for each } j = 1, 2, \ldots, M.
\end{cases}
\]

Geometric interpretation: there is no hyperplane that separates \( S_0 \) from the columns of \( D \). Such a hyperplane would have an arbitrage portfolio as a normal vector. Now according
to Farkas’ lemma, the non–existence of such a normal vector is equivalent to the existence of non–negative numbers $z_1, z_2, \ldots, z_M$ such that

$$S_0 = \sum_{j=1}^{M} z_j d^j$$

or, equivalently,

$$S_0 = Dz$$

where $z \in \mathbb{R}_+^M$. ■

A popular interpretation of the result $S_0 = Dz$ is the following (but beware of over–interpretation). Define

$$q_i = \frac{z_i}{\beta}$$

where

$$\beta = \sum_{i=1}^{M} z_i.$$

Then $q$ can be thought of as a probability distribution on $\Omega$ and we may conclude the following.

**Theorem.** The market $S$ is arbitrage–free iff there is a scalar $\beta > 0$ and probabilities (non-negative numbers that sum to 1) $q_1, q_2, \ldots, q_m$ such that

$$S_0 = \beta \mathbb{E}^Q[S_1] = \beta[q_1 S_1(\omega_1) + q_2 S_1(\omega_2) + \ldots + q_M S_1(\omega_M)]$$

and we call the numbers $\{q_i\}$ *martingale probabilities* (for reasons that are not immediately obvious in this context). Economically speaking, the $z_i = \beta q_i$ are “state prices”. Relative prices of £1 in state $i$. Arrow–Debreu prices.

Yet another approach is to define a so–called **pricing kernel**. For an arbitrage–free market $S$, there exists a (scalar) random variable $m : \Omega \to \mathbb{R}$ such that such that

$$S_0 = \mathbb{E}[m \cdot S_1]$$

where the expectation is now taken under the “objective” measure $P$. Provided the outcomes $\omega_i$ all obtain with strictly positive probabilities, we have

$$m(\omega_i) = \beta Q(\{\omega_i\})/P(\{\omega_i\}) = \beta q_i/p_i.$$
where the $p_i$s are the objective probabilities, i.e. those probabilities that actually govern the outcomes. So the pricing kernel takes care both of discounting and the change of probability measure.

### 2.2.1 Pricing contingent claims

**Definition.** A contingent claim is a mapping $X : \Omega \to \mathbb{R}$. We represent it as a vector $x \in \mathbb{R}^M$ via the formula $x_i = X(\omega_i)$. Interpretation: the contract $x$ entitles the owner to $\mathcal{E} x_i$ in state $\omega_i$.

**Theorem.** Let $S$ be an arbitrage-free market. Then there exist $\beta, q$ such that if each contingent claim $x$ is priced according to

$$
\pi_0[X] = \beta q \cdot x = \beta \mathbb{E}^Q[X]
$$

(1)

then the market consisting of all contingent claims is arbitrage free. If the market is complete (in this case: for each contingent claim $x$ there exists a portfolio $h$ such that $D^T h = x$), then the $\beta$ and the $Q$ are unique, and (1) represents the unique way of pricing each contingent claim so as to avoid arbitrage.

**Remark.** $D^T$ is the *transpose* of the matrix $D$, i.e. the matrix whose rows are the columns of $D$ and whose columns are the rows of $D$.

Notice that we can equivalently write (1) as

$$
\pi_0[X] = \mathbb{E}[m \cdot X]
$$

(2)

where $m$ is a non-negative random variable and $\mathbb{E}$ is the expected value defined by the objective probabilities. Under complete markets, this $m$ is unique.
3 Equilibrium asset pricing

In this Section I present the consumption-based capital asset pricing model (C-CAPM).\textsuperscript{1} We will be trying to account for the “equity premium”. In so doing, we will discover that volatile or unpredictable returns as such is irrelevant for the equilibrium rate of return: what matters is the covariance of payoffs with consumption.

3.1 The environment

People live for two periods. The population is constant, $N(t) = N$ and endowment profiles are constant across individuals and across time, $\omega^h_t = [\omega_1, \omega_2]$.

There are two assets in this economy: private lending, associated with a gross real interest rate of $r(t)$ and land which yields an uncertain crop $d(t)$. There are $A$ units of land available, and the crop behaves as follows: $d(t) = d + \varepsilon(t)$ where

$$\varepsilon(t) = \begin{cases} +\sigma & \text{with probability } 1/2 \\ -\sigma & \text{with probability } 1/2 \end{cases}$$

The random variables $\varepsilon(t)$ are independent.

We assume that everyone has the same utility function and that it is separable across time:

$$u^h(c^h_t(t), c^h_t(t + 1)) = u(c^h_t(t)) + \beta u(c^h_t(t + 1)).$$

We also assume that, in the presence of uncertainty, people maximize expected utility. What does this assumption amount to? Well, von Neumann and Morgenstern (1944) showed that, under some very reasonable (but apparently counterfactual—see Machina (1987)) assumptions, choice under uncertainty can be represented as maximization of expected utility.

\textsuperscript{1} This material is based on lectures notes by David Domeij at the Stockholm School of Economics and Kjetil Storesletten at the University of Oslo.
A typical individual $h$ born in period $t$ maximizes

$$E \left[ u(c^h(t)) + \beta u(c^h(t+1)) \right]$$

subject to

$$c^h_t \leq \omega_1 - \ell^h(t) - p(t)a^h(t)$$
$$c^h_t (t+1) \leq \omega_2 + r(t)\ell^h(t) + a^h(t)[p(t+1) + d(t+1)]$$

Manipulating the first-order conditions for optimal choice, we get the “fundamental asset pricing equations”.

$$\frac{1}{r(t)} = \beta E \left[ \frac{u'(c^h_t(t+1))}{u'(c^h_t(t))} \right]$$  \hspace{1cm} (3)

$$p(t) = \beta E \left\{ \frac{u'(c^h_t(t+1))}{u'(c^h_t(t))} \cdot [p(t+1) + d + \varepsilon(t+1)] \right\}$$  \hspace{1cm} (4)

Notice that the random variable $\beta u'(c^h_t(t+1))/u'(c^h_t(t))$ works as a pricing kernel here!

Since all individuals born in a given period are identical (they have the same preferences and endowments), we must have, in a (symmetric) competitive equilibrium,

$$\ell^h(t) = 0$$

and

$$a^h(t) = \frac{A}{N}.$$  

What remains to do is to find prices $r(t)$ and $p(t)$ that satisfy our fundamental asset pricing equations and also our market clearing conditions above. Since the environment is stationary, we look for a stationary equilibrium where $p(t) = p$ and $r(t) = r$. The result is

$$\frac{1}{r} = \beta E \left\{ \frac{u'\left(\omega_2 + [p + d + \varepsilon(t+1)]A/N\right)}{u'(\omega_1 - pA/N)} \right\}$$  \hspace{1cm} (5)

and

$$p = \beta E \left\{ \frac{u'\left(\omega_2 + [p + d + \varepsilon(t+1)]A/N\right)}{u'(\omega_1 - pA/N)} \cdot [p + d + \varepsilon(t+1)] \right\}$$  \hspace{1cm} (6)
3.1.1 Imposing further restrictions

In order to get sharper results, let’s make some more assumptions about preferences, crops and endowments.

3.1.1.1 Risk neutral agents  Risk neutral preferences means that the utility function is linear in consumption so that \( u'(c) = \alpha \). Then (5) and (6) become

\[
\frac{1}{r} = \beta
\]

and

\[
p = \beta(p + d).
\]

These equations say that the risk free rate \( r \) and the average rate of return on land \((p + d)/p\) are the same: \( 1/\beta \).

Alternatively, define the average equity premium via

\[
\hat{r} = \mathbb{E}\left\{ \frac{p + d + \varepsilon(t + 1)}{p} - r \right\}
\]

In the risk neutral case, this is zero.

3.1.1.2 Risk averse agents  A much more interesting case is if \( u \) is strictly concave.

Specifically, we will assume

\[
u(c) = \ln c
\]

so that

\[
u'(c) = \frac{1}{c}.
\]

To further simplify matters, assume \( \omega_2 = 0 \) and that \( A = N \).

Then our equilibrium conditions become

\[
\frac{1}{r} = \beta \mathbb{E}\left\{ \frac{\omega_1 - p}{p + d + \varepsilon(t + 1)} \right\}
\]
and

\[ p = \beta (\omega_1 - p) . \]

Substituting the second equilibrium condition into the first, we get

\[ r = \frac{p + d}{p} - \frac{\sigma^2}{p(p + d)}. \]

Solving for \( p \) from the second condition, we get

\[ p = \frac{\beta}{1 + \beta} \cdot \omega_1 \]

which can of course be used to eliminate \( p \) from the expression for \( r \), though it is a bit messy.

Notice that if \( \sigma = 0 \) then \( r = (p + d)/p \) as in the risk neutral case. Notice also that the expected equity premium is

\[ \hat{\epsilon} = \frac{(1 + \beta)^2}{\omega_1 (\beta \omega_1 + (1 + \beta)d)} \cdot \sigma^2 \]

which evidently is decreasing in \( d \) and increasing in \( \sigma \).

References


Appendix: Farkas’ lemma

Proposition (Farkas’ lemma). If $A_{m \times n}$ is a real matrix and if $b \in \mathbb{R}^m$, then exactly one of the following statements is true.

1. $Ax = b$ for some $x \in \mathbb{R}^n$

2. $(y \cdot A) \in \mathbb{R}^n_+$ and $y \cdot b < 0$ for some $y \in \mathbb{R}^m$.

Proof. In what follows we will sometimes say that $x \geq 0$ when $x$ is a vector. This means that all the components are non-negative. To prove that (1) implies that not (2), suppose that $Ax = b$ for some $x \geq 0$. Then $y \cdot Ax = y \cdot b$. But then (2) is not true. If it were, then $y \cdot A \geq 0$ and hence $y \cdot Ax \geq 0$. But then $y \cdot b \geq 0$. Hence (1) implies not (2). Next we show that not (1) implies (2). Let $X$ be the convex cone spanned by the columns $a_1, a_2, \ldots, a_n$ of $A$, i.e.

$$X = \left\{ a \in \mathbb{R}^n : a = \sum_{i=1}^{n} \lambda_i a_i; \lambda_i \geq 0, i = 1, 2, \ldots, n \right\}.$$ 

Suppose there is no $x \geq 0$ such that $Ax = b$. Then $b \notin X$. Since $X$ is closed and convex, the Separating Hyperplane Theorem says that there is a $y \in \mathbb{R}^m$ such that $y \cdot a > y \cdot b$ for all $a \in X$. Since $0 \in X$, $y \cdot 0 = 0 > y \cdot b$. Now suppose (to yield a contradiction) that not $y \cdot A \geq 0$. Then there is a column in $A$, say $a_k$, such that $y \cdot a_k < 0$. Since $X$ is a convex cone and $a_k \in X$, then $(\alpha a_k) \in X$ for all $\alpha \geq 0$. Thus for sufficiently large $\alpha$, $y \cdot (\alpha a_k) = \alpha (y \cdot a_k) < y \cdot b$. But this contradicts separation. Thus $y \cdot a^i \geq 0$ for all columns $a^i$ of $A$, i.e. $y \cdot A \geq 0$. □

Separating Hyperplane Theorem. Let $X \subseteq \mathbb{R}^n$ be closed and convex, and suppose $y \notin X$. Then there is an $a \in \mathbb{R}$ and an $h \in \mathbb{R}^n$ such that $h \cdot x > a > h \cdot y$ for all $x \in X$.

Proof. Omitted. □