1 Facts

According to Aiyagari (1993), the average annual real rate of return on 3-month U.S. Treasury bills in the post-war period has been about 1 percent. On stocks, this rate of return has been about 6 percent (4 percent in the last 200 years). Thus the equity premium was about 5 percentage points. According to Jagannathan et al. (2000), the equity premium was about 7 percentage points 1926-70, but only 0.7 of a percentage point 1970-99. See also http://pages.stern.nyu.edu/~admodar.

This set of lecture notes will develop the consumption-based capital asset pricing model (C-CAPM) which is based on the concept of competitive equilibrium. A subsequent set of lecture notes will investigate if it can account for the above facts. It can’t—unless we focus on the last 30 years only.

2 Prequel to C-CAPM: no-arbitrage pricing

How far can we get in determining the “correct” price of assets by just assuming that there are no arbitrage opportunities? Notice that this is a weaker assumption than (competitive) equilibrium. Surprisingly far, as it turns out. In particular, we can work out how to price derivatives—claims that are defined in terms of other securities, e.g. options.
2.1 A two-period model

Let $t = 0, 1$ (today and tomorrow). There are $N$ securities. The price of security $n$ at time $t$ is denoted by $S_t^n$. We write

$$ S_t = \begin{bmatrix} S_1^t \\ S_2^t \\ \vdots \\ S_N^t \end{bmatrix}. $$

$S_0$ is deterministic, but $S_1$ is stochastic. We write

$$ S_1(\omega) = \begin{bmatrix} S_1^1(\omega) \\ S_1^2(\omega) \\ \vdots \\ S_1^N(\omega) \end{bmatrix} $$

where $\omega \in \Omega = \{\omega_1, \omega_2, \ldots, \omega_M\}$.

Now define the matrix $D$ via

$$ D_{N \times M} = \begin{bmatrix} S_1^1(\omega_1) & S_1^1(\omega_2) & \ldots & S_1^1(\omega_M) \\ S_1^2(\omega_1) & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \vdots \\ S_1^N(\omega_1) & \ldots & \ldots & S_1^N(\omega_M) \end{bmatrix} = \begin{bmatrix} d_1^1 & d_2^1 & \ldots & d_M^1 \\ \vdots & \vdots & \ddots & \vdots \\ \vdots & \vdots & \ddots & \vdots \\ d_1^M & \ldots & \ldots & d_M^M \end{bmatrix}.$$ 

Definition. A portfolio is a vector $h \in \mathbb{R}^N$. Interpretation: $h_n$ is the number of securities of type $n$ purchased at $t = 0$.

Remark: Fractional holdings as well as short positions ($h_n < 0$) are allowed.

The value of a portfolio $h$ at time $t$ is given by

$$ V_t(h) = \sum_{n=1}^{N} h_n S_t^n = h^T S_t. $$

Definition. A vector $h \in \mathbb{R}^N$ is called an arbitrage portfolio if

$$ V_0(h) < 0 $$
and

\[ V_1(h) \geq 0 \]

for all \( \omega \in \Omega \).

**Remark.** We can weaken the condition “for all \( \omega \in \Omega \)” to “with probability one” if we want — but in this context that wouldn’t add much.

**Theorem.** Let securities prices \( S \) be as above. Then there exists no arbitrage portfolio iff there exists a \( z \in \mathbb{R}^M_+ \) such that

\[ S_0 = Dz. \]

**Remark.** This means that today’s (period 0’s) price vector has to lie in the convex cone spanned by tomorrow’s (period 1’s) possible prices vectors. (A convex cone is a subset \( C \) of a vector space \( X \) such that for any \( x, y \in C \) and \( \alpha \geq 0 \) we have \( \alpha x \in C \) and \( x + y \in C \).)

**Proof.** Absence of arbitrage opportunities means that the following system of inequalities has no solution for \( h \).

\[
\begin{cases}
    h^T S_0 < 0 \\
    h^T d^j \geq 0 \text{ for each } j = 1, 2, \ldots, M.
\end{cases}
\]

Geometric interpretation: there is no hyperplane that separates \( S_0 \) from the columns of \( D \). Such a hyperplane would have an arbitrage portfolio as a normal vector. Now according to Farkas’ lemma, the non–existence of such a normal vector is equivalent to the existence of non–negative numbers \( z_1, z_2, \ldots, z_M \) such that

\[ S_0 = \sum_{j=1}^{M} z_j d^j \]

or, equivalently,

\[ S_0 = Dz \]

where \( z \in \mathbb{R}^M_+ \). ■

A popular interpretation of the result \( S_0 = Dz \) is the following (but beware of over–interpretation). Define

\[ q_i = \frac{z_i}{\beta} \]

where

\[ \beta = \sum_{i=1}^{M} z_i. \]
Then $q$ can be thought of as a probability distribution on $\Omega$ and we may conclude the following.

**Theorem.** The market $S$ is arbitrage–free iff there is a scalar $\beta > 0$ and probabilities (non-negative numbers that sum to 1) $q_1, q_2, \ldots, q_m$ such that

$$
S_0 = \beta E^Q[S_1] = \beta[q_1 S_1(\omega_1) + q_2 S_1(\omega_2) + \ldots + q_M S_1(\omega_M)]
$$

and we call the numbers $\{q_i\}$ martingale probabilities (for reasons that are not immediately obvious in this context). Economically speaking, the $q_i$s are “state prices”. Relative prices of $\$1$ in state $i$. Arrow–Debreu prices.

Yet another approach is to define a so–called **pricing kernel**. For an arbitrage–free market $S$, there exists a (scalar) random variable $m : \Omega \to \mathbb{R}$ such that such that

$$
S_0 = E[m \cdot S_1]
$$

where the expectation is now taken under the “objective” measure $P$. Provided the outcomes $\omega_i$ all obtain with strictly positive probabilities, we have

$$
m(\omega_i) = \beta Q(\{\omega_i\})/P(\{\omega_i\}) = \beta q_i/p_i.
$$

where the $p_i$s are the objective probabilities. So the pricing kernel takes care both of discounting and the change of probability measure.

### 2.1.1 Pricing contingent claims

**Definition.** A contingent claim is a mapping $X : \Omega \to \mathbb{R}$. We represent it as a vector $x \in \mathbb{R}^M$ via the formula $x_i = X(\omega_i)$. Interpretation: the contract $x$ entitles the owner to $\$x_i$ in state $\omega_i$.

**Theorem.** Let $S$ be an arbitrage–free market. Then there exist $\beta, q$ such that if each contingent claim $x$ is priced according to

$$
\pi_0[X] = \beta q^T x = \beta E^Q[X]
$$

then the market consisting of all contingent claims is arbitrage free. If the market is complete (in this case: for each contingent claim $x$ there exists a portfolio $h$ such that $D^T h = x$), then the $\beta$ and the $Q$ are unique, and (1) represents the unique way of pricing each contingent claim so as to avoid arbitrage.
Notice that we can equivalently write (1) as
\[
\pi_0[X] = [m \cdot X]
\] (2)
where \(m\) is a non-negative random variable. Under complete markets, this \(m\) is unique.

3 Equilibrium asset pricing

In this Section I present the consumption-based capital asset pricing model (C-CAPM).\(^1\) We will be trying to account for the “equity premium”. In so doing, we will discover that volatile or unpredictable returns as such is irrelevant for the equilibrium rate of return: what matters is the covariance of payoffs with consumption.

3.1 The environment

People live for two periods. The population is constant, \(N(t) = N\) and endowment profiles are constant across individuals and across time, \(\omega^h_t = [\omega_1, \omega_2]\).

There are two assets in this economy: private lending, associated with a gross real interest rate of \(r(t)\) and land which yields an uncertain crop \(d(t)\). There are \(A\) units of land available, and the crop behaves as follows: \(d(t) = d + \varepsilon_t\) where
\[
\varepsilon_t = \begin{cases} 
+\sigma & \text{with probability } 1/2 \\
-\sigma & \text{with probability } 1/2 
\end{cases}
\]
The random variables \(\varepsilon_t\) are independent.

We assume that everyone has the same utility function and that it is separable across time:
\[
u^h(c^h_t(t), c^h_t(t + 1)) = u(c^h_t(t)) + \beta u(c^h_t(t + 1)).
\]
We also assume that, in the presence of uncertainty, people maximize expected utility. What does this assumption amount to? Well, von Neumann and Morgenstern (1944) showed that,

\(^1\) This material is based on lectures notes by David Domeij at the Stockholm School of Economics and Kjetil Storesletten at the University of Oslo.
under some very reasonable (but apparently counterfactual—see Machina (1987)) assumptions, choice under uncertainty can be represented as maximization of expected utility.

A typical individual $h$ born in period $t$ maximizes

$$E \left[ u(c^h_t(t)) + \beta u(c^h_{t+1}(t+1)) \right]$$

subject to

$$c^h_t(t) \leq \omega_1 - \ell^h(t) - p(t)a^h(t)$$
$$c^h_t(t+1) \leq \omega_2 + r(t)\ell^h(t)$$

Manipulating the first-order conditions for optimal choice, we get the “fundamental asset pricing equations”.

$$\frac{1}{r(t)} = \beta E \left[ \frac{u'(c^h_{t+1}(t+1))}{u'(c^h_t(t))} \right]$$

$$p(t) = \beta E \left\{ \frac{u'(c^h_{t+1}(t+1))}{u'(c^h_t(t))} \cdot [p(t+1) + d + \varepsilon(t+1)] \right\}$$

Notice that the random variable $\beta u'(c^h_t(t+1))/u'(c^h_t(t))$ works as a pricing kernel here!

Since all individuals born in a given period are identical (they have the same preferences and endowments), we must have, in a (symmetric) competitive equilibrium,

$$\ell^h(t) = 0$$

and

$$a^h(t) = \frac{A}{N}.$$ 

What remains to do is to find prices $r(t)$ and $p(t)$ that satisfy our fundamental asset pricing equations and also our market clearing conditions above. Since the environment is stationary, we look for a stationary equilibrium where $p(t) = p$ and $r(t) = r$. The result is

$$\frac{1}{r} = \beta E \left\{ \frac{u'(\omega_2 + [p + d + \varepsilon(t+1)]A/N)}{u'(\omega_1 - pA/N)} \right\}$$

and

$$p = \beta E \left\{ \frac{u'(\omega_2 + [p + d + \varepsilon(t+1)]A/N)}{u'(\omega_1 - pA/N)} \cdot [p + d + \varepsilon(t+1)] \right\}$$
3.1.1 Imposing further restrictions

In order to get sharper results, let’s make some more assumptions about preferences, crops and endowments.

3.1.1.1 Risk neutral agents  Risk neutral preferences means that the utility function is linear in consumption so that $u'(c) = \alpha$. Then (5) and (6) become

$$\frac{1}{r} = \beta$$

and

$$p = \beta [p + d].$$

These equations say that the risk free rate $r$ and the average rate of return on land $(p + d)/p$ are the same: $1/\beta$.

Alternatively, define the average equity premium via

$$\hat{r} = \mathbb{E} \left\{ \frac{p + d + \varepsilon(t + 1)}{p} - r \right\}.$$

In the risk neutral case, this is zero.

3.1.1.2 Risk averse agents  A much more interesting case is if $u$ is strictly concave. Specifically, we will assume

$$u(c) = \ln c$$

so that

$$u'(c) = \frac{1}{c}.$$

To further simplify matters, assume $\omega_2 = 0$ and that $A = N$.

Then our equilibrium conditions become

$$\frac{1}{r} = \beta \mathbb{E} \left\{ \frac{\omega_1 - p}{[p + d + \varepsilon(t + 1)]} \right\}$$

and

$$p = \beta (\omega_1 - p).$$
Solving for $p$ from the second condition, we get

$$p = \frac{\beta}{1 + \beta} \cdot \omega_1.$$

Substituting this into the first equilibrium condition, we get

$$r = \frac{p + d}{p} - \frac{\sigma^2}{p(p + d)}.$$

Notice that if $\sigma = 0$ then $r = (p + d)/p$ as in the risk neutral case. Notice also that the expected equity premium is

$$\hat{r} = \frac{(1 + \beta)^2}{\omega_1(\beta \omega_1 + (1 + \beta)d)} \cdot \sigma^2$$

which evidently is decreasing in $d$ and increasing in $\sigma$.

References


Appendix: Farkas’ lemma

**Proposition (Farkas’ lemma).** If $A \in \mathbb{R}^{m \times n}$ is a real matrix and if $b \in \mathbb{R}^m$, then exactly one of the following statements is true.

1. $Ax = b$ for some $x \in \mathbb{R}^n$

2. $(y^T A) \in \mathbb{R}_+^n$ and $y^T b < 0$ for some $y \in \mathbb{R}^m$.

**Proof.** In what follows we will sometimes say that $x \geq 0$ when $x$ is a vector. This means that all the components are non-negative. To prove that (1) implies that not (2), suppose that $Ax = b$ for some $x \geq 0$. Then $y^T Ax = y^T b$. But then (2) is not true. If it were, then $y^T A \geq 0$ and hence $y^T Ax \geq 0$. But then $y^T b \geq 0$. Hence (1) implies not (2). Next we show that not (1) implies (2). Let $X$ be the convex cone spanned by the columns $a^1, a^2, ..., a^n$ of $A$, i.e.

$$X = \left\{ a \in \mathbb{R}^n : a = \sum_{i=1}^{n} \lambda_i a^i; \lambda_i \geq 0, i = 1, 2, \ldots, n \right\}.$$ 

Suppose there is no $x \geq 0$ such that $Ax = b$. Then $b \notin X$. Since $X$ is closed and convex, the *Separating Hyperplane Theorem* says that there is a $y \in \mathbb{R}^m$ such that $y^T a > y^T b$ for all $a \in X$. Since $0 \in X$, $y^T 0 > y^T b$ and consequently $y^T b < 0$. Now suppose (to yield a contradiction) that not $y^T A \geq 0$. Then there is a column in $A$, say $a^k$, such that $y^T a^k < 0$. Since $X$ is a convex cone and $a^k \in X$, we have $(\alpha a^k) \in X$ for all $\alpha \geq 0$. But by the (absurd) supposition, for sufficiently large $\alpha$, we have $y^T (\alpha a^k) = \alpha (y^T a^k) < y^T b$, and this contradicts separation so the supposition cannot be true. Thus $y^T a^i \geq 0$ for all columns $a^i$ of $A$, i.e. $y^T A \geq 0$. ■

**Separating Hyperplane Theorem.** Let $X \subset \mathbb{R}^n$ be closed and convex, and suppose $y \notin X$. Then there is an $a \in \mathbb{R}$ and an $h \in \mathbb{R}^n$ such that $h^T x > a > h^T y$ for all $x \in X$.

**Proof.** Omitted. ■