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Welfare theorems with infinitely many goods

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1 Introduction

Welfare theorems are concerned with the conditions under which decentralized decision making can lead to a socially optimal outcome, or at least one that cannot be unambiguously improved upon by making everybody better off. The idea that a market economy (i.e. an economy based on private ownership, mutually agreed exchange, and competition) achieves a socially desirable outcome, is often attributed to Smith (1776). In fact, it was expressed a bit earlier than that in Chydenius (1765). What we'll be discussing here is a mathematical model of an idealized competitive economy, establishing two theorems. The first claims that any competitive equilibrium is Pareto efficient; the second that any Pareto efficient allocation is part of some competitive equilibrium. This mathematical approach was pioneered by Walras (1874) and perfected by Debreu (1959).

2 The first welfare theorem

Our commodity space will be a normed vector space X. Prices will be represented by a continuous linear functional $\varphi : X \to \mathbb{R}$.

Let X be a normed vector space. Let there be I consumers and J producers. Each consumer i has a non-empty set of options $X_i \subset X$. If $x_i \in X_i$ we say that x_i is **available** (but not necessarily affordable) to i. Each producer j has a non-empty set $Y_j \subset X$. If $y_j \in Y_j$ we say that y_j is available to j.

A **feasible allocation** is a profile $\mathcal{X} = (x_1, x_2, \ldots, x_I)$ and a profile $\mathcal{Y} = (y_1, y_2, \ldots, y_J)$ such that $x_i \in X_i$ for all $i = 1, 2, \ldots, I$, $y_j \in Y_j$ for all $j = 1, 2, \ldots, J$ and

$$\sum_{i=1}^{I} x_i = \sum_{j=1}^{J} y_j.$$

A **feasible consumption allocation** is a profile $\mathcal{X} = (x_1, x_2, \ldots, x_I)$ such that $x_i \in X_i$ for all $i = 1, 2, \ldots, I$ and there exists a profile $\mathcal{Y} = (y_1, y_2, \ldots, y_J)$ such that $y_j \in Y_j$ for all $j = 1, 2, \ldots, J$ and

$$\sum_{i=1}^{I} x_i = \sum_{j=1}^{J} y_j.$$

Each consumer i comes equipped with a utility function $u_i : X_i \to \mathbb{R}$.

We will say that consumer i's choice $x_i \in X_i$ **maximizes utility** given the pricing function φ if it is such that $\varphi(x) > \varphi(x_i)$ for all $x \in X_i$ such that $u_i(x) > u_i(x_i)$. Equivalently, $u_i(x) > u_i(x_i)$ and $x \in X_i$ together imply $\varphi(x) > \varphi(x_i)$.

Notice that there is no need to raise the question whether a consumer is spending all his wealth (or more). We simply *define* the wealth of consumer i as $\varphi(x_i)$.

We say that producer j **maximizes profits** given φ if his choice $y_j \in Y_j$ is such that $\varphi(y_j) \geq \varphi(y)$ for all $y \in Y_j$. In other words, if $\varphi(y) > \varphi(y_j)$ then $y \notin Y_j$.

A **competitive equilibrium** is a feasible allocation $(\mathcal{X}, \mathcal{Y})$ and a continuous linear functional φ such that x_i maximizes utility for each $i = 1, 2, \ldots, I$ and y_j maximizes profits for all $j = 1, 2, \ldots, J$.

What we want to do is to show that an allocation that is Pareto superior to a competitive equilibrium allocation is not feasible. We do this in a couple of steps.

Consider a competitive equilibrium $(\mathcal{X}^*, \mathcal{Y}^*, \varphi)$ and a Pareto superior alternative allocation $(\mathcal{X}, \mathcal{Y})$. By definition, $u_i(x_i) \geq u_i(x_i^*)$ for all i and $u_i(x_i) > u_i(x_i^*)$ for some i, say $i = k$. By utility maximization, $\varphi(x_k) > \varphi(x_k^*)$. But it is not obvious that $\varphi(x_i) \geq \varphi(x_i^*)$ for all i. For that we assume local non-satiation of preferences.

A utility function u_i is said to exhibit **local non-satiation** if for any $x \in X_i$ and any $\varepsilon > 0$ there is a $y \in X_i$ such that $\|x - y\| < \varepsilon$ and $u_i(y) > u_i(x)$.

Now assume that $u_i(x_i) \geq u_i(x_i^*)$ and that u_i exhibits local non-satiation. We want to show that $\varphi(x_i) \geq \varphi(x_i^*)$. So suppose $\varphi(x_i) < \varphi(x_i^*)$, hoping that this will lead to a contradiction. By the continuity of φ, there exists an $\varepsilon > 0$ such that for all $y \in X_i$ such that $\|x_i - y\| < \varepsilon$ we have $\varphi(y) < \varphi(x_i^*)$. By local non-satiation, one such y is such that $u_i(y) > u_i(x_i) \geq u_i(x_i^*)$. Hence, by utility maximization, $\varphi(y) > \varphi(x_i^*)$, a contradiction.

Thus we may conclude that

$$\sum_{i=1}^{I} \varphi(x_i) > \sum_{i=1}^{I} \varphi(x_i^*) =$$

$$\varphi\left(\sum_{i=1}^{I} x_i^*\right) = \varphi\left(\sum_{j=1}^{J} y_j^*\right) = \sum_{j=1}^{J} \varphi(y_j^*)$$

Because $\mathcal{Y}^*$ maximizes profits, we know that

$$\sum_{j=1}^{J} \varphi(y_j^*) \geq \sum_{j=1}^{J} \varphi(y_j)$$

and we may conclude that

$$\varphi\left(\sum_{i=1}^{I} x_i\right) > \varphi\left(\sum_{j=1}^{J} y_j\right)$$

and hence that

$$\sum_{i=1}^{I} x_i \neq \sum_{j=1}^{J} y_j.$$

We now have a proof of...

Theorem 1 (First Welfare Theorem) *Let $(\mathcal{X}, \mathcal{Y}, \varphi)$ be a competitive equilibrium and suppose the utility functions exhibit local non-satiation. Then the allocation $(\mathcal{X}, \mathcal{Y})$ is Pareto optimal.*

3 The second welfare theorem

For the purpose of proving the second welfare theorem, we will say that consumer i's choice $x_i \in X_i$ **minimizes costs** given the pricing function φ if it is such that $\varphi(x) \geq \varphi(x_i)$ for all $x \in X_i$ such that $u_i(x) \geq u_i(x_i)$. Equivalently, $u_i(x) \geq u_i(x_i)$ and $x \in X_i$ together imply $\varphi(x) \geq \varphi(x_i)$.

In this context, the fact that our definition of utility maximization (or cost minimization) makes no reference to endowments or wealth is even less problematic. If you insist on having an endowment profile, just make the endowment profile equal to the given allocation!

Theorem 2 (The Hahn-Banach Separation Theorem) *Let X be a normed vector space. Let $A \subset X$ and $B \subset X$ be non-empty and convex. Let A be open. Suppose $A \cap B = \varnothing$. Then there exists a continuous linear functional $\varphi : X \to \mathbb{R}$ and a constant $\alpha \in \mathbb{R}$ such that*

$$\varphi(x) > \alpha \geq \varphi(y)$$

for all $x \in A$ and all $y \in B$.

Proof. See this document or this one. ■

Consider now a Pareto optimal allocation $(\mathcal{X}^*, \mathcal{Y}^*)$. This means that it is feasible and that there is no feasible Pareto superior alternative.

For each i, define $A_i \subset X_i$ via

$$A_i = \{x \in X_i : u_i(x) > u_i(x_i^*)\}.$$

We would like A_i to be non-empty, convex and open. For that purpose, we assume (local) non-satiation, convexity and continuity of preferences. Local non-satiation evidently guarantees that A_i is not empty.

We say that $u_i : X_i \to \mathbb{R}$ **represents convex preferences** if X_i is convex and if whenever $u_i(x) \geq u_i(y)$ and $0 \leq \lambda \leq 1$ we have $u_i(\lambda x + (1-\lambda)y) \geq u_i(y)$.

We say that u_i is **continuous** if for any open set $O \subset \mathbb{R}$ the set $\{x \in X_i : u_i(x) \in O\}$ is an open subset of X. (Note that it has to be open as a subset of X, not just as a subset of X_i!)

Now define $A \subset X$ via

$$A = \{x \in X : x = \sum_{i=1}^{I} \widetilde{x}_i \text{ and } \widetilde{x}_i \in A_i\}$$

and it is not hard to verify that A is non-empty, convex and open.

Next, define the production set Y via

$$Y = \{y \in X : y = \sum_{j=1}^{J} y_j \text{ and } y_j \in Y_j\}.$$

We want this set, too, to be non-empty and convex. So we just go ahead and assume that it is convex. (We have already assumed that the Y_js are non-empty.)

By Pareto optimality of $(\mathcal{X}^*, \mathcal{Y}^*)$, A and Y are disjoint. Hence they satisfy the premisses of Theorem 2 and we have a constant α and a continuous linear functional φ such that

$$\varphi(x) > \alpha \geq \varphi(y)$$

for all $x \in A$ and $y \in Y$. Our job now is to show that $\mathcal{X}^*$ maximizes utility and that $\mathcal{Y}^*$ maximizes profits given φ.

First we show that for any profile $\mathcal{X}$ such that $u_i(x_i) \geq u_i(x_i^*)$ we have

$$\varphi\left(\sum_{i=1}^{I} x_i\right) \geq \alpha.$$

Local non-satiation means for any $\varepsilon > 0$ there is an $\widetilde{x}_i$ such that $\|\widetilde{x}_i - x_i\| < \varepsilon$ and $u_i(\widetilde{x}_i) > u_i(x_i)$ and hence $\widetilde{x}_i \in A_i$. Defining

$$\widetilde{x} = \sum_{i=1}^{I} \widetilde{x}_i,$$

it is clear that $\widetilde{x} \in A$ and hence that $\varphi(\widetilde{x}) > \alpha$. Taking limits as $\varepsilon \to 0$, we get the desired result.

Notice that, annoyingly, we don't exploit strict separation. Is there any way of doing that in order to simplify the proof?

Incidentally, we may apply our result to $\mathcal{X}^*$ itself. Thus

$$\varphi\left(\sum_{i=1}^{I} x_i^*\right) \geq \alpha.$$

On the other hand, $\sum_{i=1}^{I} x_i^* = \sum_{j=1}^{J} y_j^*$ which is a member of Y by feasibility. Hence

$$\varphi\left(\sum_{i=1}^{I} x_i^*\right) \leq \alpha$$

and of course it follows that

$$\varphi\left(\sum_{i=1}^{I} x_i^*\right) = \varphi\left(\sum_{j=1}^{J} y_j^*\right) = \alpha.$$

Next, we show that if x_i is such that $u_i(x_i) \geq u_i(x_i^*)$ then $\varphi(x_i) \geq \varphi(x_i^*)$. Apparently

$$\varphi\left(x_i + \sum_{k \neq i} x_k^*\right) \geq \alpha = \varphi\left(x_i^* + \sum_{k \neq i} x_k^*\right)$$

and the result follows.

Of course, this isn't quite what we wanted to show. This is cost minimization, not utility maximization. To show utility maximization, we want to show that if $x_i \in X_i$ and $u_i(x_i) > u_i(x_i^*)$ then $\varphi(x_i) > \varphi(x_i^*)$. We already know that $\varphi(x_i) \geq \varphi(x_i^*)$. Could it be that $\varphi(x_i) = \varphi(x_i^*)$? Well, yes, unless we make a further assumption.

We say that the profile $(\mathcal{X}, \mathcal{Y}, \varphi)$ has the **cheaper-bundle property** if for each i there is an $x \in X_i$ such that $\varphi(x) < \varphi(x_i)$.

Now suppose, for a contradiction, that $\varphi(x_i) = \varphi(x_i^*)$. By the cheaper-bundle property, there is an $x_i' \in X_i$ such that $\varphi(x_i') < \varphi(x_i^*)$. By convexity of X_i we have a bundle

$x_i'' = \lambda x_i + (1-\lambda)x_i'$ such that $\varphi(x_i'') < \varphi(x_i^*)$. By continuity of u_i, if λ is sufficiently close to 1 we have $u_i(x_i'') > u_i(x_i^*)$. This is inconsistent with cost minimization.

Finally, if $y_j \in Y_j$ we have

$$\varphi\left(y_j + \sum_{k\neq j} y_k^*\right) \leq \alpha = \varphi\left(y_j^* + \sum_{k\neq j} y_k^*\right) \leq \alpha$$

and so $\mathcal{Y}^*$ maximizes profits.

We have now proven...

Theorem 3 (Second welfare theorem) *Suppose $(\mathcal{X}^*, \mathcal{Y}^*)$ is Pareto optimal, that u_i are continuous, exhibit local non-satiation and represent convex preferences. Suppose the production set Y is convex. Then there is a φ such that $\mathcal{X}^*$ minimizes costs and $\mathcal{Y}^*$ maximizes profits. If, in addition, $(\mathcal{X}^*, \mathcal{Y}^*, \varphi)$ has the cheaper-bundle-property, then $\mathcal{X}^*$ maximizes utility.*

3.1 Inner product representation of continuous linear functionals

Continuous linear functionals are nice, but it would be even better if they could be written as inner products in some sense. For Hilbert spaces, we have a very nice representation theorem.

Theorem 4 (Riesz representation) *Let $(\mathscr{H}, (\cdot,\cdot))$ be a Hilbert space and let $\varphi\colon \mathscr{H} \to \mathbb{R}$ be a continuous linear functional. Then there is a unique $y \in \mathscr{H}$ such that $\varphi(x) = (x, y)$ for all $x \in \mathscr{H}$.*

Proof. Consider the sets

$$M = \{x \in \mathscr{H} : \varphi(x) = 0\}$$

and

$$M^\perp = \{x \in \mathscr{H} : (x, y) = 0 \text{ for all } y \in M\}.$$

Evidently $M \cap M^\perp = \{0\}$. Meanwhile, $M^\perp$ is at most one-dimensional. To see this, we will show that any two members of $M^\perp$ are linearly dependent. So let $x \in M^\perp$, $w \in M^\perp$. Suppose that at least one of them is not also a member of M so we do not have $\varphi(x) = \varphi(w) = 0$. (If there is no such pair x, w then $M^\perp \subset M$ and consequently $M^\perp = \{0\}$.) Apparently

$$\varphi(\varphi(x)w - \varphi(w)x) = 0.$$

Hence $(\varphi(x)w - \varphi(w)x) \in M$. But it is also a member of $M^\perp$, since $M^\perp$ is a vector space (why?). Hence

$$\varphi(x)w - \varphi(w)x = 0$$

but we assumed that $\varphi(x) = \varphi(w) = 0$ was not the case, so x and w must be linearly dependent. Thus there exists a $y_0 \in M^\perp$ such that

$$M^\perp = \{x \in \mathscr{H} : x = \alpha y_0 \text{ for some } \alpha \in \mathbb{R}\}.$$

If $y_0 = 0$, then evidently $M = \mathscr{H}$ and we can set $y = 0$. So suppose $y_0 \neq 0$. We are now going to choose y as a member of $M^\perp$. What does y have to be so that

$$\varphi(y_0) = (y_0, y)?$$

Since $y \in M^\perp$, all we have to do is solve for α the following equation.

$$\varphi(y_0) = (y_0, \alpha y_0) = \alpha \|y_0\|^2.$$

Thus

$$y = \alpha y_0.$$

where

$$\alpha = \frac{\varphi(y_0)}{\|y_0\|^2}.$$

But then (why?) we have

$$\varphi(x) = (x, y)$$

for all $x \in M^\perp$. Finally, by the projection theorem, any $z \in \mathscr{H}$ can be written as

$$z = x + w$$

with $x \in M$ and $w \in M^{\perp}$. Evidently $\varphi(z) = \varphi(x) + \varphi(w) = \varphi(w)$ because φ is linear and $x \in M$. Meanwhile, $(z, y) = (x + w, y) = (w, y) = \varphi(w)$ because $x \in M$ and $y \in M^{\perp}$ and because, as we just established, $\varphi(w) = (w, y)$ for all $w \in M^{\perp}$. So we have $\varphi(z) = (z, y)$ for all $z \in \mathscr{H}$. That takes care of existence. To show uniqueness, suppose $y_1 \in \mathscr{H}$ and $y_2 \in \mathscr{H}$ are such that

$$(x, y_1) = (x, y_2)$$

for all $x \in \mathscr{H}$. This of course means that

$$(x, y_1 - y_2) = 0$$

for all $x \in \mathscr{H}$. In particular,

$$(y_1 - y_2, y_1 - y_2) = 0$$

and hence $y_1 - y_2 = 0$. ■

An example is the commodity space ℓ^2, defined as the set of sequences of real numbers $x = \{x_t\}_{t=0}^{\infty}$ with

$$\sum_{t=0}^{\infty} x_t^2 < \infty$$

with inner product

$$(x, y) = \sum_{t=0}^{\infty} x_t y_t.$$

and norm

$$\|x\| = \sqrt{\sum_{t=0}^{\infty} x_t^2}.$$

By the Riesz representation theorem, any continuous linear functional $\varphi : \ell^2 \to \mathbb{R}$ has the representation

$$\varphi(x) = \sum_{t=0}^{\infty} p_t x_t$$

for some sequence p such that

$$\sum_{t=0}^{\infty} p_t^2 < \infty.$$

Another example is static but involves uncertainty. Let $\mathcal{L}^2(\Omega, \mathcal{F}, \mathsf{P})$ be the set of contingent claims $X : \Omega \to \mathbb{R}$ such that

$$\mathsf{E}[X^2] < \infty$$

with inner product

$$(X, Y) = \mathsf{E}[XY].$$

Now let $\varphi : \mathcal{L}^2 \to \mathbb{R}$ be a continuous linear functional. Then Riesz' theorem guarantees that there is a square integrable random variable $L : \Omega \to \mathbb{R}$ such that

$$\varphi(X) = \mathsf{E}[LX].$$

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