Lecture 3: The neoclassical growth model

In these notes we apply our knowledge of intertemporal budget constraints and dynamic optimization to analyze the neoclassical growth model.

1 Prequel: an economy with exogenous saving

The Solow (1956) model was set in continuous time, and there are some definite advantages of that. In particular, it enables us to solve the model explicitly if technology is Cobb-Douglas, and a number of useful insights stem from that.

Recall from intermediate macroeconomics the differential equation describing the evolution of the capital stock, assuming Cobb-Douglas technology.

\[ \dot{k}(t) = sk^\alpha(t) - \delta k(t) \]

where \( s \) is the fraction of output saved (as opposed to consumed). This differential equation is of the Bernoulli type, and can be solved explicitly. Let’s do it! We begin by introducing the auxiliary variable \( y \), defined via

\[ y(t) := k^{1-\alpha}(t). \]
Given this definition, we have, by the chain rule,

\[ \dot{y}(t) = (1 - \alpha)k^{-\alpha}(t)\dot{k}(t). \]

Alternatively, we may write

\[ \dot{k}(t) = \frac{1}{1 - \alpha} \cdot k^\alpha(t) \cdot \dot{y}(t). \]

Substituting this into Solow’s differential equation, we have

\[ \frac{1}{1 - \alpha} \cdot k^\alpha(t) \cdot \dot{y}(t) = sk^\alpha(t) - \delta k(t). \]

Multiplying by \(1 - \alpha\) and dividing by \(k^\alpha\) (assuming, as we should, that \(k > 0\)), we get

\[ \dot{y}(t) = (1 - \alpha)s - \delta(1 - \alpha)y(t). \]

This is a linear differential equation, and it can be solved in two steps. First we find a “particular” solution which in this case is the steady state. Setting \(\dot{y} = 0\), we have

\[ y^* = \frac{s}{\delta}. \]

Now notice that

\[ \dot{y}(t) = -\delta(1 - \alpha)[y(t) - y^*]. \]

It follows that

\[ y(t) - y^* = \exp\{-\delta(1 - \alpha)t\} \cdot [y(0) - y^*]. \]

Hence

\[ k(t) = \left\{ \exp\{-\delta(1 - \alpha)t\} \cdot \left[k^{\frac{1}{1-\alpha}}(0) - \frac{s}{\delta}\right] + \frac{s}{\delta}\right\}^{\frac{1}{1-\alpha}}. \]

Notice that, as expected, the solution does converge to the steady state as long as \(k(0) > 0\). Notice also that the rate of convergence to the steady state increases in the depreciation rate \(\delta\) and decreases in the capital share \(\alpha\).
2 A closed economy in continuous time

Suppose we endogenize the savings rate. Suppose a social planner maximizes

\[ \int_0^\infty e^{-\rho t} \ln(c(t)) dt \]

subject to \( k(0) = k_0 > 0 \) given, \( k(t) \geq 0, c(t) \geq 0 \) and

\[ \dot{k}(t) = f(k(t)) - c(t). \] (1)

The log specification of utility is of course a bit special, but in general we would like to capture the assumption that people like their consumption smooth.

Using Pontryagin’s maximum principle, or something like it (see Lang, 1993), we may characterize the solution via the following differential equation.

\[ \dot{c}(t) = [f'(k(t)) - \rho]c(t). \]

What this means intuitively is that consumption should increase over time if it pays to postpone consumption. Notice that \( \rho \) is a measure of impatience, i.e. a measure of the cost of postponing consumption. The marginal product of capital \( f'(k(t)) \), on the other hand, is a measure of how much we gain by postponing consumption (by forgoing consumption today and instead investing more). If the cost of postponing consumption exceeds its benefits, consumption should gradually decline. And vice versa.

Combining this differential equation with Equation (1), we have a two-dimensional system of differential equations. We may characterize its solution with the help of a phase diagram.
3 A closed economy with exogenous labour supply

Suppose a social planner maximizes

$$\sum_{t=0}^{\infty} \beta^t u(c_t)$$

subject to $c_t \geq 0$, $k_t \geq 0$ and

$$c_t + k_{t+1} = (1 - \delta)k_t + Af(k_t)$$

for $t = 0, 1, \ldots$

We have solved this for $\delta = 1$, $f(k) = k^\alpha$. Now we consider what to do if all of these
conditions are not met.

Let’s begin by characterizing the solution. We start by constructing a Lagrangian.

\[ \mathcal{L} = \sum_{t=0}^{\infty} \beta^t u(c_t) + \sum_{t=0}^{\infty} \beta^t \lambda_t [(1 - \delta)k_t + Af(k_t) - c_t - k_{t+1}] \]

The optimality conditions are

\[ u_{c,t} - \lambda_t = 0, \]
\[ -\lambda_t + \beta Af_{k,t+1} \lambda_{t+1} = 0. \]

The first equation enables us to eliminate the Lagrange multiplier.

\[ u_{c,t} = \beta u_{c,t+1} Af_{k,t+1}. \]

This equation can be interpreted as \( MRS = MRT \) for consumption at different dates. But what do we do with it? Use numerical methods. See Lecture 4 and Lecture 5.

4  A closed economy with endogenous labour supply

Suppose a social planner maximizes

\[ \sum_{t=0}^{\infty} \beta^t u(c_t, \ell_t) \]

subject to \( \ell_t \geq 0, \ell_t = 1 - h_t, k_t \geq 0 \) and

\[ c_t + k_{t+1} = (1 - \delta)k_t + Af(k_t, h_t) \]

for \( t = 0, 1, \ldots \)

Let’s again construct a Lagrangian.

\[ \mathcal{L} = \sum_{t=0}^{\infty} \beta^t u(c_t, 1 - h_t) + \sum_{t=0}^{\infty} \beta^t \lambda_t [(1 - \delta)k_t + Af(k_t, h_t) - c_t - k_{t+1}] . \]
The optimality conditions are

\[-u_{t,t} + \lambda_t A f_{h,t} = 0,\]
\[u_{c,t} - \lambda_t = 0,\]
\[-\lambda_t + \beta A f_{k,t+1} \lambda_{t+1} = 0.\]

The second equation enables us to eliminate the Lagrange multiplier. We are then left with

\[u_{t,t} = u_{c,t} A f_{h,t}\]

and

\[u_{c,t} = \beta u_{c,t+1} A f_{k,t+1}.\]

Both of these equations can be interpreted as MRS = MRT, one for leisure and consumption, the other for consumption at different dates. Again, we solve them (approximately) using numerical methods.

5 An open economy with a collateral constraint

A small open economy without any frictions jumps immediately to its steady state. To make it interesting, we introduce a friction. In this section, we follow the approach of Barro et al. (1995).

Suppose a small open economy faces a world market interest rate of \(R\) but that its net foreign asset position is constrained.

A social planner maximizes

\[\sum_{t=0}^{\infty} \beta^t u(c_t)\]

subject to \(a_0\) and \(k_0\) given and

\[a_{t+1} + k_{t+1} + c_t = Ra_t + f(k_t)\]
and
\[ a_{t+1} + \varphi k_{t+1} \geq 0. \]

Form the Lagrangean
\[
L = \sum_{t=0}^{\infty} \beta^t u(c_t) + \sum_{t=0}^{\infty} \beta^t \lambda_t [Ra_t + f(k_t) - a_{t+1} - k_{t+1} - c_t] + \sum_{t=0}^{\infty} \beta^t \mu_t [a_{t+1} + \varphi k_{t+1}]
\]
where the minus sign is there to ensure that \( \mu_t \geq 0 \). The first-order conditions are
\[
u'(c_t) - \lambda_t = 0 \quad (c)
\]
for \( t = 0, 1, \ldots \),
\[-\lambda_t + \beta \lambda_{t+1} f'(k_{t+1}) + \varphi \mu_t = 0 \quad (k_{t+1})
\]
for \( t = 0, 1, \ldots \), and
\[-\lambda_t + \beta R \lambda_{t+1} + \mu_t = 0 \quad (a_{t+1})
\]
for \( t = 0, 1, \ldots \)

Notice that if the collateral constraint does not bind so that \( \mu_t = 0 \), we get
\[-\lambda_t + \beta \lambda_{t+1} f'(k_{t+1}) = 0
\]
and
\[-\lambda_t + \beta \lambda_{t+1} R = 0
\]
so that \( f'(k_{t+1}) = R \). If the collateral constraint binds so that \( \mu_t > 0 \), we may conclude that \( f'(k_{t+1}) > R \). The argument goes as follows. Subtracting Equation \((a_{t+1})\) from Equation \((k_{t+1})\), we get
\[
\beta \lambda_{t+1} [f'(k_{t+1}) - R] = \mu_t (1 - \varphi).
\]

Suppose now that \( \varphi < 1 \); otherwise it immediately follows that \( \mu_t = 0 \) and so the collateral constraint does not bind. The result follows.

In practice, at least if \( R = 1/\beta \), it turns out that \( \mu_t \to 0 \) as \( t \to \infty \), so that the
collateral constraint does not bind in the long run.

How do we deal with inequality conditions numerically? The standard approach is to use a so-called complementarity method. Instead of saying that \( \mu_t \geq 0 \) and \( a_{t+1} + \varphi k_{t+1} \geq 0 \) with equality in at least one case, we insist that

\[
\min\{\mu_t, a_{t+1} + \varphi k_{t+1}\} = 0
\]

for \( t = 0, 1, \ldots \)

6 An open economy with quadratic adjustment costs

Suppose that labour supply is endogenous (which is not important in this context) and that, at the firm level, it is costly to change the capital stock so that per period profits are given by

\[
\pi_t = Af(k_t, h_t) - w_t h_t - (r_t + \delta) k_t - \omega \cdot \Omega(k_t, k_{t-1})
\]

where

\[
\Omega(x, y) = \frac{(x - y)^2}{x + y}.
\]

Notice that this specification preserves constant returns to scale so that neither the number of firms in a given country, nor the distribution of capital between them, matters. For the record, we have

\[
\begin{align*}
\Omega_x &= \frac{x^2 + 2xy - 3y^2}{(x + y)^2}, \\
\Omega_y &= \frac{-3x^2 + 2xy + y^2}{(x + y)^2}, \\
\Omega_{xx} &= \frac{8y^2}{(x + y)^3}, \\
\Omega_{yy} &= \frac{8x^2}{(x + y)^3}
\end{align*}
\]
and

\[ \Omega_{xy} = -\frac{8xy}{(x+y)^3}. \]

The sum of discounted profits is given by

\[ \Pi_0 = \sum_{t=0}^{\infty} \left( \prod_{s=0}^{t} (1 + r_s)^{-1} \right) \pi_t \tag{4} \]

Profit maximization still implies

\[ w_t = Af_{h,t} \]

but the condition for capital becomes more complicated. Specifically,

\[ r_t = Af_{k,t} - \delta - \omega \cdot \Omega_x(k_t, k_{t-1}) - \frac{\beta u_{c,t+1}}{u_{c,t}} \cdot \omega \cdot \Omega_y(k_{t+1}, k_t) \tag{5} \]

where we have replaced the discount factor \((1 + r_{t+1})^{-1}\) with the marginal rate of intertemporal substitution \(\beta u_{c,t+1}/u_{c,t}\) so as to avoid an infinite recursion.
References

